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Stephen Ward , Paul Fannon
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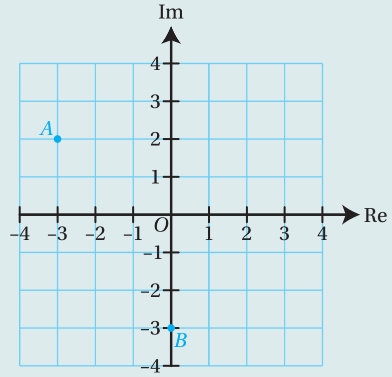
1 Further complex numbers: powers and roots

In this chapter you will learn how to:

- raise complex numbers to integer powers (De Moivre's theorem)
- work with complex exponents
- find roots of complex numbers
- use roots of unity
- find quadratic factors of polynomials
- use a relationship between complex number multiplication and geometric transformations.

Before you start…

Further Mathematics Student Book 1, Chapter 1	You should know how to find the modulus and argument of a complex number.	1 Find the modulus and argument of $-3 + 4\mathrm{i}$.
Further Mathematics Student Book 1, Chapter 1	You should be able to represent complex numbers on an Argand diagram.	2 Write down the complex numbers corresponding to the points A and B. 
Further Mathematics Student Book 1, Chapter 1	You should know how to work with complex numbers in Cartesian form.	3 Given that $z = 3 - 2\mathrm{i}$ and $w = 2 + \mathrm{i}$, evaluate: a $z - w$ b $\frac{z}{w}$.
Further Mathematics Student Book 1, Chapter 1	You should be able to multiply and divide complex numbers in modulus–argument form.	4 Given that $z = 10\left(\cos\frac{3\pi}{4} + \mathrm{i}\sin\frac{3\pi}{4}\right)$ and $w = 2\left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right)$, find: a zw b $\frac{z}{w}$. Give the arguments in the range $(-\pi, \pi]$.

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Further Mathematics Student Book 1, Chapter 1	You should be able to work with complex conjugates.	5 Write down the complex conjugate of: a $5\text{i} - 3$ b $3\left(\cos\frac{\pi}{4} + \text{i}\sin\frac{\pi}{4}\right)$.
Further Mathematics Student Book 1, Chapter 1	You should know how to relate operations with complex numbers to transformations on an Argand diagram.	6 Let $a = 2 + \text{i}$ and z be any complex number. Describe a geometrical transformation that maps: a z to z^* b z to $z + a$.

Extending arithmetic with complex numbers

You already know how to perform basic operations with complex numbers, both in Cartesian and in modulus–argument forms. Modulus–argument form is particularly well suited to multiplication and division. In this chapter you will see how you can use this to find powers and roots of complex numbers. This chapter also includes a definition of complex powers that can make calculations even simpler.

You will also meet roots of unity, which are the solutions of the equation $z^n = 1$. They have some useful algebraic and geometric properties. Some of the applications include finding exact values of trigonometric functions.

Because you can represent complex numbers as points on an Argand diagram, operations with complex numbers have a geometric interpretation. You can use this fact to solve some problems that at first sight have nothing to do with complex numbers. This is just one example of the use of complex numbers to solve real-life problems.

Rewind

You met complex numbers in Further Mathematics Student Book 1, Chapter 1.

Fast forward

You will learn more about links between complex numbers and trigonometry in Chapter 2.

Section 1: De Moivre's theorem

In Further Mathematics Student Book 1, Chapter 1, you learnt that you can write complex numbers in Cartesian form, $x + \text{i}y$, or in modulus–argument form, $r(\cos\theta + \text{i}\sin\theta)$. You also learnt the rules for multiplying complex numbers in modulus–argument form:

$$|zw| = |z||w| \text{ and } \arg(zw) = \arg z + \arg w$$

You can apply this result to find powers of complex numbers. If a complex number has modulus r and argument θ, then multiplying $z \times z$ gives that z^2 has modulus r^2 and argument 2θ. Repeating this process, you can see that

$$|z^n| = |z|^n \text{ and } \arg(z^n) = n\arg z$$

In other words, when you raise a complex number to a power, you raise the modulus to the same power and multiply the argument by the power.

Key point 1.1

De Moivre's theorem

For a complex number, z, with modulus r and argument θ:

$$z^n = (r(\cos\theta + \mathrm{i}\sin\theta))^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta)$$

for every integer power n.

For positive integer powers, you can prove this result by induction.

Rewind

Proof by induction was covered in Further Mathematics Student Book 1, Chapter 12. For Proof 1 you will also need the compound angle formulae from A Level Mathematics Student Book 2, Chapter 8.

Focus on...

See Focus on ... Proof 1 for a proof that extends De Moivre's theorem to all rational n.

PROOF 1

When $n = 1$:
$(r(\cos\theta + \mathrm{i}\sin\theta))^1 = r(\cos\theta + \mathrm{i}\sin\theta)$
so the result is true for $n = 1$.

> Check that the result is true for $n = 1$.

Assuming that the result is true for some k:
$(r(\cos\theta + \mathrm{i}\sin\theta))^k = r^k(\cos k\theta + \mathrm{i}\sin k\theta)$

> Assume that the result is true for some k and write down what that means.

Then for $n = k + 1$:
$(r(\cos\theta + \mathrm{i}\sin\theta))^{k+1} = r^k(\cos k\theta + \mathrm{i}\sin k\theta) \times r(\cos\theta + \mathrm{i}\sin\theta)$

> Make a link between $n = k$ and $n = k + 1$. In this case use $z^{k+1} = z^k z$.

$= r^{k+1}(\cos k\theta\cos\theta + \mathrm{i}\cos k\theta\sin\theta + \mathrm{i}\sin k\theta\cos\theta - \sin k\theta\sin\theta)$

> Group real and imaginary parts.

$= r^{k+1}((\cos k\theta\cos\theta - \sin k\theta\sin\theta) + \mathrm{i}(\cos k\theta\sin\theta + \sin k\theta\cos\theta))$
$= r^{k+1}(\cos(k+1)\theta + \mathrm{i}\sin(k+1)\theta)$

> Use:
> $\cos(A + B) = \cos A\cos B - \sin A\sin B$
> and
> $\sin(A + B) = \sin A\cos B + \cos A\sin B$

Hence the result is true for $n = k + 1$.

> This is the result you are trying to prove, but with n replaced by $k + 1$.

The result is true for $n = 1$, and if it is true for some k then it is also true for $k + 1$. Therefore it is true for all n integers $\geqslant 1$ by induction.

> Remember to write the full conclusion.

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You can use De Moivre's theorem to evaluate powers of complex numbers.

WORKED EXAMPLE 1.1

Evaluate, without a calculator, $\frac{(1+i)^{10}}{4i}$.

$|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$

$\arg(1+i) = \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4}$

$\therefore\ 1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

First find the modulus and argument of each number.

$|4i| = 4,\ \arg(4i) = \frac{\pi}{2}$

$\therefore\ 4i = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

By De Moivre's theorem:

$$\left(\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^{10} = (\sqrt{2})^{10}\left(\cos\frac{10\pi}{4} + i\sin\frac{10\pi}{4}\right)$$

$$= 2^5\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right)$$

$$= 32\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

The argument needs to be between $-\pi$ and π:

$\frac{5\pi}{2} - 2\pi = \frac{\pi}{2}$

Dividing the moduli and subtracting the arguments:

$$\frac{(1+i)^{10}}{4i} = \frac{32\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)}{4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)}$$

$$= \frac{32}{4}(\cos 0 + i\sin 0)$$

$$= 8$$

WORK IT OUT 1.1

Evaluate $\left(1+i\sqrt{3}\right)^6$.

Which is the correct solution? Identify the errors made in the incorrect solutions.

Solution 1

$$\left(1+i\sqrt{3}\right)^6 = 1 + 6\left(i\sqrt{3}\right)^1 + 15\left(i\sqrt{3}\right)^2 + 20\left(i\sqrt{3}\right)^3 + 15\left(i\sqrt{3}\right)^4 + 6\left(i\sqrt{3}\right)^5 + \left(i\sqrt{3}\right)^6$$

$$= 1 + 6i\sqrt{3} - 45 - 60i\sqrt{3} + 135 + 54i\sqrt{3} - 27$$

$$= 64$$

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Solution 2	$1+\mathrm{i}\sqrt{3} = 2\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}$ So, using De Moivre's theorem: $\left(1+\mathrm{i}\sqrt{3}\right)^6 = 2^6\cos\left(\frac{\pi}{6}\times 6\right) + \mathrm{i}\sin\left(\frac{\pi}{6}\times 6\right)$ $= 64\cos\pi + \mathrm{i}\sin\pi$ $= -64$
Solution 3	$\left(1+\mathrm{i}\sqrt{3}\right)^6 = 1^6 + \left(\mathrm{i}\sqrt{3}\right)^6$ $= 1+27$ $= 28$

You can also prove that De Moivre's theorem works for negative integer powers.

WORKED EXAMPLE 1.2

Let $z = r(\cos\theta + \mathrm{i}\sin\theta)$.

a Find the modulus and argument of $\frac{1}{z}$.

b Hence prove De Moivre's theorem for negative integer powers.

a Multiplying top and bottom by the complex conjugate:

$$\frac{1}{z} = \frac{1}{r(\cos\theta + \mathrm{i}\sin\theta)} \times \frac{(\cos\theta - \mathrm{i}\sin\theta)}{(\cos\theta - \mathrm{i}\sin\theta)}$$

$$= \frac{\cos\theta - \mathrm{i}\sin\theta}{r(\cos^2\theta + \sin^2\theta)}$$

$$= \frac{1}{r}(\cos\theta - \mathrm{i}\sin\theta)$$

Use $\cos^2\theta + \sin^2\theta = 1$

$$= \frac{1}{r}(\cos(-\theta) + \mathrm{i}\sin(-\theta))$$

To find the modulus and argument, you need to write the number in this form. Remember that $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$.

Hence $\left|\frac{1}{z}\right| = \frac{1}{r}$ and $\arg\left(\frac{1}{z}\right) = -\theta$.

This means that you can write $\frac{1}{z} = \frac{1}{r}(\cos(-\theta) + \mathrm{i}\sin(-\theta))$.

b

You need to prove that $z^{-n} = r^{-n}(\cos(-n\theta) + \mathrm{i}\sin(-n\theta))$.

Using De Moivre's theorem for positive powers:

$$\left(\frac{1}{z}\right)^n = \left(\frac{1}{r}(\cos(-\theta) + \mathrm{i}\sin(-\theta))\right)^n$$

$$= \left(\frac{1}{r}\right)^n(\cos(-n\theta) + \mathrm{i}\sin(-n\theta))$$

Since you have already proved De Moivre's theorem for positive powers, you can use $z^{-n} = (z^{-1})^n = \left(\frac{1}{z}\right)^n$ with the modulus and argument of $\frac{1}{z}$ found in part **a**.

Hence $z^{-n} = r^{-n}(\cos(-n\theta) + \mathrm{i}\sin(-n\theta))$, as required.

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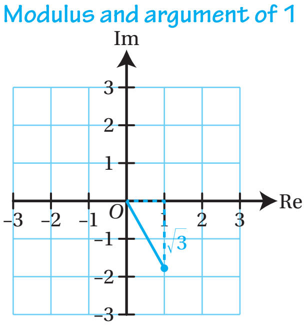
WORKED EXAMPLE 1.3

Find the modulus and argument of $\frac{1}{(1-i\sqrt{3})^7}$.

Modulus and argument of $1 - i\sqrt{3}$:

The best way to find the modulus and argument is to sketch a diagram.



$\sqrt{1^2+\sqrt{3}^2} = 2$

$\tan^{-1}\frac{\sqrt{3}}{1} = \frac{\pi}{3}$

$z = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$

Applying De Moivre's theorem for negative powers:

$z^{-7} = 2^{-7}\left(\cos\left(-7\times-\frac{\pi}{3}\right) + i\sin\left(-7\times-\frac{\pi}{3}\right)\right)$

$= \frac{1}{128}\left(\cos\frac{7\pi}{3} + i\sin\frac{7\pi}{3}\right)$

This is in the form $r(\cos\theta + i\sin\theta)$ so you can just read off the modulus and the argument.

The modulus is $\frac{1}{128}$ and the argument is $\frac{7\pi}{3} - 2\pi = \frac{\pi}{3}$.

The argument needs to be between 0 and 2π, so you need to take away 2π.

EXERCISE 1A

1 Evaluate each expression, giving your answer in the form $r(\cos\theta + i\sin\theta)$.

a **i** $\left(2\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)\right)^6$ **ii** $\left(3\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)\right)^4$

b **i** $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^3$ **ii** $\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)^4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^2$

c **i** $\frac{\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^6}{\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^3}$ **ii** $\frac{\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^2}{\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^6}$

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2 Given that $z = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$:

a write z^2, z^3 and z^4 in the form $r(\cos\theta + i\sin\theta)$

b represent z, z^2, z^3 and z^4 on the same Argand diagram.

3 **a** Given that $z = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$:

i write z^2, z^3 and z^4 in modulus-argument form

ii represent z, z^2, z^3 and z^4 on the same Argand diagram.

b For which natural numbers n is $z^n = z$?

4 **a** Find the modulus and argument of $1 + i\sqrt{3}$.

Hence, clearly showing your working, find:

b $(1 + i\sqrt{3})^5$ in modulus-argument form.

c $(1 + i\sqrt{3})^5$ in Cartesian form.

5 **a** Write $-\sqrt{2} + i\sqrt{2}$ in the form $r(\cos\theta + i\sin\theta)$.

b Hence, clearly showing your working, find $(-\sqrt{2} + i\sqrt{2})^6$ in simplified Cartesian form.

6 Find the smallest positive integer value of n for which $\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)^n$ is real.

7 Find the smallest positive integer value of k such that $\left(\cos\frac{3\pi}{28} + i\sin\frac{3\pi}{28}\right)^k$ is pure imaginary.

Section 2: Complex exponents

The rules for multiplying complex numbers in modulus–argument form look just like the rules of indices.

Compare

$$r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2) = r_1r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$

with

$$k_1e^{x_1} \times k_2e^{x_2} = k_1k_2e^{x_1+x_2}.$$

You can extend the definition of powers to imaginary numbers so that all the rules of indices still apply.

Key point 1.2

Euler's formula:

$$e^{i\theta} \equiv \cos\theta + i\sin\theta$$

Did you know?

Substituting $\theta = \pi$ into Euler's formula and rearranging gives $e^{i\pi} + 1 = 0$. This equation, called Euler's identity, connects five important numbers from different areas of mathematics. It is often cited as 'the most beautiful' equation in mathematics.

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The Maclaurin series you know for e^x, $\sin x$ and $\cos x$ provide a justification for Euler's formula if you accept that the series for e^x converges for imaginary as well as for real numbers.

You can write a complex number with modulus r and argument θ in **exponential form** as $re^{i\theta}$. You now have two different ways of writing complex numbers with a given modulus and argument.

Rewind

See Further Mathematics Student Book 1, Chapter 11, for a reminder of the Maclaurin series. You will meet these again in Chapter 10.

Key point 1.3

$$re^{i\theta} = r(\cos\theta + i\sin\theta)$$

When working with complex numbers in exponential form you can use all the normal rules of indices.

WORKED EXAMPLE 1.4

Given that $z = 2e^{\frac{i\pi}{12}}$ and $w = \frac{1}{2}e^{\frac{i\pi}{4}}$, find z^5w^3 in the form $x + iy$.

$z^5w^3 = \left(2e^{\frac{i\pi}{12}}\right)^5\left(\frac{1}{2}e^{\frac{i\pi}{4}}\right)^3$ — You can do all the calculations in exponential form and then convert to Cartesian form at the end.

$= 2^5e^{\frac{5i\pi}{12}} \times \frac{1}{2^3}e^{\frac{3i\pi}{4}}$ — Use rules of indices for the powers: $\frac{5}{12} + \frac{3}{4} = \frac{7}{6}$

$= 4e^{\frac{7i\pi}{6}}$

$= 4\left(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right)$ — Now write in terms of trigonometric functions and evaluate.

$= 4\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$

$= -2\sqrt{3} - 2i$

You can combine Euler's formula with rules of indices to raise any real number to any complex power.

WORKED EXAMPLE 1.5

Find, correct to three significant figures, the value of:

a e^{2+3i} **b** 3^{2+3i}.

a $e^{2+3i} = e^2e^{3i}$ — Use rules of indices to separate the real and imaginary parts of the power.

$= e^2(\cos 3 + i\sin 3)$ — Use Euler's formula for the imaginary power.

$= -7.32 + 1.04i$ — Expand the brackets and give the answer to 3 s.f. Remember that the arguments of the trigonometric functions are in radians.

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b $3^{2+3i} = (e^{\ln 3})^{2+3i}$

$= e^{2\ln 3}e^{(3\ln 3)i}$

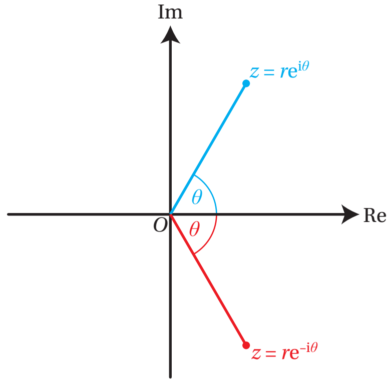
$= 9(\cos(\ln 27) + i\sin(\ln 27))$

$= -8.89 - 1.38i$

You only know how to raise e to a complex power, so express 3 as a power of e.

Use rules of indices and then Euler's formula. Note that $e^{2\ln 3} = e^{\ln 9} = 9$ and $3\ln 3 = \ln 27$.

The complex conjugate of a number is easy to find when written in exponential form. This is best seen on an Argand diagram, where taking the complex conjugate is represented by a reflection in the real axis. In this case it is best to take the argument between $-\pi$ and π.



Fast forward

You will use the exponential form of complex numbers when solving second-order differential equations in Chapter 11.

Key point 1.4

The complex conjugate of $z = re^{i\theta}$ is $z^* = re^{-i\theta}$.

Tip

Note that if $r = 1$, you also have $z^* = \frac{1}{z} = e^{-i\theta}$.

In Further Mathematics Student Book 1, Chapter 2, you used complex conjugates when solving polynomial equations.

WORKED EXAMPLE 1.6

A cubic equation has real coefficients and two of its roots are 1 and $2e^{\frac{i\pi}{3}}$. Find the equation in the form $x^3 + ax^2 + bx + c = 0$.

The roots are: $1, 2e^{\frac{i\pi}{3}}$ and $2e^{-\frac{i\pi}{3}}$.

Complex roots occur in conjugate pairs, so you can write down the third root.

$-a = 1 + 2e^{\frac{i\pi}{3}} + 2e^{-\frac{i\pi}{3}}$

$= 1 + 4\cos\frac{\pi}{3}$

$= 3$

$\therefore a = -3$

Use the formulae for sums and products of roots to find the coefficients of the equation: $-a = x_1 + x_2 + x_3$

Remember that $z + z^* = 2\text{Re}(z)$, and that $\text{Re}\left(e^{\frac{i\pi}{3}}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$

$b = \left(1 \times 2e^{\frac{i\pi}{3}}\right) + \left(1 \times 2e^{-\frac{i\pi}{3}}\right) + \left(2e^{\frac{i\pi}{3}} \times 2e^{-\frac{i\pi}{3}}\right)$

$= 4\cos\left(\frac{\pi}{3}\right) + 4$

$= 6$

$b = x_1x_2 + x_2x_3 + x_3x_1$

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$-c = 1 \times 2e^{\frac{i\pi}{3}} \times 2e^{-\frac{i\pi}{3}} = 4$ ...... $-c = x_1x_2x_3$

$\therefore\ c = -4$

Hence the equation is $x^3 - 3x^2 + 6x - 4 = 0$.

EXERCISE 1B

You can use your calculator to perform operations with complex numbers in Cartesian, modulus-argument and exponential forms, as well as to convert from one form to another. Do the questions in this exercise without a calculator first, then use a calculator to check your answers.

1 Write each complex number in Cartesian form without using trigonometric functions.

a **i** $3e^{i\frac{\pi}{6}}$ **ii** $4e^{\frac{i\pi}{4}}$

b **i** $4e^{i\pi}$ **ii** $5e^{2\pi i}$

c **i** $e^{\frac{2\pi i}{3}}$ **ii** $2e^{\frac{3\pi}{2}i}$

2 Write each complex number in the form $re^{i\theta}$.

a **i** $5 + 5i$ **ii** $2\sqrt{3} - 2i$

b **i** $-\frac{1}{2} + \frac{1}{2}i$ **ii** $1 - i\sqrt{3}$

c **i** $-4i$ **ii** -5

3 Write the answer to each calculation in the form $re^{i\theta}$.

a **i** $4e^{i\frac{\pi}{6}} \times 5e^{i\frac{\pi}{4}}$ **ii** $\dfrac{5e^{i\frac{3\pi}{4}}}{10e^{i\frac{\pi}{4}}}$

b **i** $\dfrac{\left(2e^{i\frac{\pi}{4}}\right)^3}{\left(5e^{i\frac{\pi}{3}}\right)^2}$ **ii** $\dfrac{2e^{i\frac{\pi}{3}}}{\left(e^{i\frac{\pi}{6}}\right)^5}$

4 Represent each complex number on an Argand diagram.

a **i** $e^{i\frac{\pi}{3}}$ **ii** $e^{i\frac{3\pi}{4}}$

b **i** $5e^{i\frac{\pi}{2}}$ **ii** $2e^{-i\frac{\pi}{2}}$

5 Let $z = 2e^{\frac{i\pi}{12}}$ and $w = 4e^{\frac{i\pi}{3}}$. Show that $z^2 + w = 2(1 + i)(1 + \sqrt{3})$.

6 Let $z = 2e^{\frac{i\pi}{3}}$ and $w = 3e^{\frac{-i\pi}{6}}$. Write each complex number in the form $x + iy$.

a $\dfrac{z}{w}$ **b** z^5w^3.

7 Write e^{4+3i} in the form $x + iy$, where x and y are real, giving your answer correct to three significant figures.

8 Write $e^{2-i\frac{\pi}{3}}$ in exact Cartesian form.